Practice §12.3: Dot Product

Instructions: Complete each of the following exercises for practice.

1. Which of the expressions below are meaningful, and which are meaningless? Explain.

   (a) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$
   (b) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
   (c) $|\mathbf{u}|(\mathbf{v} \cdot \mathbf{w})$
   (d) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$
   (e) $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$
   (f) $|\mathbf{u}| \cdot (\mathbf{v} + \mathbf{w})$

2. Compute $\mathbf{u} \cdot \mathbf{v}$ for each pair of vectors below.

   (a) $\mathbf{u} = \langle 5, -2, 3 \rangle, \;\; \mathbf{v} = \langle 7, 1, 2 \rangle$
   (b) $\mathbf{u} = \langle p, -p, 2p \rangle, \;\; \mathbf{v} = \langle 2q, q, -q \rangle$
   (c) $|\mathbf{u}| = 7, \;\; |\mathbf{v}| = 4, \;\; \theta = 30°$
   (d) $|\mathbf{u}| = 80, \;\; |\mathbf{v}| = 50, \;\; \theta = \frac{3\pi}{4}$

3. Find the exact angle between the given vectors $\mathbf{u}$ and $\mathbf{v}$.

   (a) $\mathbf{u} = \langle 1, -4, 1 \rangle, \;\; \mathbf{v} = \langle 0, 2, -2 \rangle$
   (b) $\mathbf{u} = \langle -1, 3, 4 \rangle, \;\; \mathbf{v} = \langle 5, 2, 1 \rangle$
   (c) $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \;\; \mathbf{v} = 2\mathbf{i} - \mathbf{k}$
   (d) $\mathbf{u} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \;\; \mathbf{v} = \mathbf{j} + 2\mathbf{k}$

4. Determine whether the given vectors are orthogonal, parallel, or neither.

   (a) $\mathbf{u} = \langle 4, 5, -2 \rangle, \;\; \mathbf{v} = \langle 3, -1, 5 \rangle$
   (b) $\mathbf{u} = \langle -5, 4, -2 \rangle, \;\; \mathbf{v} = \langle 3, 4, -1 \rangle$
   (c) $\mathbf{u} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}, \;\; \mathbf{v} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$
   (d) $\mathbf{u} = 9\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}, \;\; \mathbf{v} = -6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

5. Compute the acute angle between the curves below at their points of intersection (the angle between two curves at an intersection point is the angle between the tangent lines at that point).

   (a) $f(x) = x^2, \;\; g(x) = x^3$
   (b) $f(x) = \sin(x), \;\; g(x) = \cos(x)$ for $0 \leq x \leq \frac{\pi}{2}$

6. Find the direction cosines and direction angles of the vector exactly.

   (a) $\langle 2, 1, 2 \rangle$
   (b) $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$
   (c) $\langle c, c, c \rangle$ for $c > 0$

7. Suppose vector $\mathbf{v}$ has direction angles $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$. What is the direction angle γ for $\mathbf{v}$?

8. Find scalar and vector projections of $\mathbf{v}$ onto $\mathbf{u}$.

   (a) $\mathbf{u} = \langle 4, 7, -4 \rangle, \;\; \mathbf{v} = \langle 3, -1, 1 \rangle$
   (b) $\mathbf{u} = \langle -1, 4, 8 \rangle, \;\; \mathbf{v} = \langle 12, 1, 2 \rangle$
   (c) $\mathbf{u} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \;\; \mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
   (d) $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \;\; \mathbf{v} = 5\mathbf{i} - \mathbf{k}$

9. Find the work done by constant force $\mathbf{F} = \langle 8, -6, 9 \rangle$ moving an object from point $P = (0, 10, 8)$ to $Q = (6, 12, 20)$ in a straight line, where force is measured in newtons and distance in meters.

10. A sled is pulled along a level path through snow by a rope. A 30 pound force acts at an angle of 40° above the horizontal and moves the sled 80 feet. Compute the work done by the force.

11. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors. Prove that if $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{u} - \mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ have the same length.

12. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. Prove the *Parallelogram Law*: $|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2$.

13. Suppose $\mathbf{u} \neq \mathbf{0} \neq \mathbf{v}$ and let $\mathbf{w} = |\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}$. Show that $\mathbf{w} \neq \mathbf{0}$ implies $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

14. This question walks you through a proof of the Triangle Inequality for vectors. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors.

    (a) Prove the *Cauchy-Schwarz Inequality*: $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$ (*Hint*: What is $\mathbf{u} \cdot \mathbf{v}$ geometrically?).
    (b) Prove the *Triangle Inequality*: $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ (*Hint*: Use the Cauchy-Schwarz Inequality and $|\mathbf{u} + \mathbf{v}|^2$).